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Spatially chaotic configurations and functional equations with rescaling

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Abstract. The functional equation

$$y(qt) = \frac{1}{4q}[y(t+1) + y(t-1) + 2y(t)] \qquad (0 < q < 1) \qquad t \in \mathbb{R}$$

is associated with the appearance of spatially chaotic structures in amorphous (glassy) materials. Continuous compactly supported solutions of the above equation are of special interest. We shall show that there are no such solutions for $0 < q < \frac{1}{2}$, whereas such a solution exists for almost all $\frac{1}{2} < q < 1$. The words 'for almost all q' in the previous sentence cannot be omitted. There are exceptional values of q in the interval $[\frac{1}{2}, 1]$ for which there are no integrable solutions. For example, $q = (\sqrt{5} - 1)/2 \approx 0.618$, which is the reciprocal of the 'golden ratio' is such an exceptional value. More generally, if λ is any Pisot–Vijayaraghavan number, or any Salem number, then $q = \lambda^{-1}$ is an exceptional value.

1. Introduction and formulation of the main result

1.1. Physical background

Since the establishment of the KAM theory (Arnold and Avez 1968, Moser 1973) for Hamiltonian dynamics and the discovery of strange attractors by Ruelle and Takens (1971) for dissipative dynamics, nonlinear dynamics has become an intensively studied field in physics. One of the most interesting features of nonlinear dynamics is the existence of chaotic motion which can occur e.g. in Hamiltonian systems with more than one degree of freedom. Although Hamiltonian equations of motion are deterministic, there exist initial points $x(0) = (r_1(0), \ldots, r_s(0); p_1(0), \ldots, p_s(0)) \in \mathbb{R}^{2s}$ of finite Lebesgue measure in the phase space of a particle system with s degrees of freedom for which the trajectories $x(t, x(0))$ are chaotic in time. The chaotic behaviour is best described for two-dimensional, nonlinear maps T:

$$T : (x_n, y_n) \to (x_{n+1}, y_{n+1}) \qquad n \in \mathbb{Z} \tag{1}$$

with $(x_n, y_n) \in \mathbb{R}^2$. This corresponds to a dynamical system with discrete time n. The chaotic behaviour of the dynamics defined by (1) originates from the embedding of the Bernoulli shift (Moser 1973), i.e. there exist:

(i) an alphabet $A = \{a_1, a_2, \ldots\}$ (which depends on T)

(ii) a domain $G \subset \mathbb{R}^2$ and an embedding

$$\varphi : G \to S \tag{2}$$

where S is the space of doubly-infinite sequences σ of symbols $\sigma_n \in A$

$$S = \{\sigma = (\ldots \sigma_{-2}, \sigma_{-1}, \sigma_0, \sigma_1, \sigma_2 \ldots) | \sigma_n \in A\} \tag{3}$$

(iii) a symbolic dynamical system (Bernoulli shift)

$$s : \sigma \in S \to s(\sigma) =: \sigma' \in S \tag{4}$$

where

$$\sigma'_m := (\sigma')_m = \sigma_{m-1}$$

such that the original dynamics T restricted to the domain $G \subset \mathbb{R}^2$ can be obtained from the symbolic dynamics as follows:

$$T|G = \varphi^{-1} \circ s \circ \varphi. \tag{5}$$

Equation (5) is probably one of the most important results for nonlinear chaotic dynamics. Choosing for the initial sequence $\sigma^{(0)}$ a random sequence of symbols $\sigma_n \in A$, the symbolic dynamics s generates new sequences $\sigma^{(\nu)} = s^\nu(\sigma^{(0)})$, $\nu \in \mathbb{Z}$ which are random and which result in a chaotic dynamics of T due to (5). One of us (Reichert and Schilling 1985, Schilling 1992) has recently shown that chaotic behaviour is not restricted to temporal phenomena, but may also occur in space. This spatial chaos has been used in order to interpret the existence of amorphous structures in solid materials. Let us consider a system of N particles with coordinates $r_n \in \mathbb{R}, n = 1, 2, \ldots, N$. The particles will interact with each other. The corresponding potential energy is $V(r_1, r_2, \ldots, r_N)$. A special type of configurations $r^{(\alpha)} = (r_1^{(\alpha)}, r_2^{(\alpha)}, \ldots, r_N^{(\alpha)})$, $\alpha = 1, 2, \ldots, M(N)$ are the solutions of

$$\frac{\partial V}{\partial r_n}(r) = 0 \tag{6}$$

i.e. $r^{(\alpha)}$ are stationary configurations. For a certain class of functions V it was proven (Reichert and Schilling 1985, Schilling 1992) that for $N = \infty$ there exists an alphabet $A = \{+, -\}$ such that there is a one-to-one correspondence between $r^{(\alpha)}$ and all doubly-infinite sequences $\sigma = (\ldots \sigma_{-2}, \sigma_{-1}, \sigma_0, \sigma_1, \sigma_2 \ldots), \sigma_n \in A$. This result means that the particle positions in a stationary configuration $r^{(\alpha)}$ can be obtained from σ:

$$r_n^{(\alpha)} = (f(\sigma))_n \tag{7}$$

where σ depends on α. Choosing again a random sequence σ, results in a spatially chaotic configuration $r^{(\alpha)}$ given by (7).

Since many physical quantities, e.g. the distance between particles, excitation energies, etc depend on the coordinates $r_n^{(\alpha)}$, these quantities are uniquely specified by σ due to (7). For a certain class of potentials V (Reichert and Schilling 1985, Schilling 1992), one finds a linear relationship. Let Δ_n be one of these local physical quantities. Then it has been shown that

$$\Delta_n(r^{(\alpha)}) = \sum_{j=1}^{\infty} \eta^j (\sigma_{n+j} - \sigma_{n-j-1}) \qquad n \in \mathbb{Z} \tag{8}$$

where $0 < \eta < 1$ and $\sigma_n = \sigma_n(\alpha)$.

Here a comment is in order. The condition $0 < \eta < 1$ follows from the energetic stability of the model considered and for that special case the one-to-one correspondence between $r^{(\alpha)}$ and $\sigma(\alpha)$ only holds for $0 < \eta < \frac{1}{3}$. For more details see Reichert and Schilling (1985) and Schilling (1992). Since there is no general reason why this must be fulfilled for other models, we will discuss the full range $0 < \eta < 1$, for which $\Delta_n(r^{(\alpha)})$ is finite.

Since almost all sequences σ are neither periodic nor almost periodic, the values Δ_n for $n \in \mathbb{Z}$ will be distributed in a non-trivial manner. For given α, which determines σ uniquely, the corresponding distribution function is

$$P_\sigma(\Delta) = \lim_{N\to\infty} \frac{1}{2N} \sum_{n=-N}^{N} \delta(\Delta - \Delta_n(\sigma)) \tag{9}$$

where $\Delta_n(\sigma)$ is defined by the right-hand side of (8). For a generic sequence σ where the symbols $+$ and $-$ occur with probability $\frac{1}{2}$, it has been assumed (Schilling 1992) that there exists a kind of 'ergodicity' (also called 'self-averaging' in physics) which leads to

$$P_\sigma(\Delta) = \sum_\sigma w(\sigma)\delta(\Delta - \Delta_0(\sigma)) =: P(\Delta) \tag{10}$$

for almost all σ. The probability $w(\sigma)$ is given by

$$w(\sigma) = \prod_n w_0(\sigma_n)$$

with

$$w_0(\sigma_n) \equiv \tfrac{1}{2} \qquad \text{for } \sigma_n \in \{+, -\}. \tag{11}$$

Since σ_n is interpreted now as a random variable it is easy to derive a *functional equation* for $P(\Delta)$ which is (Schilling 1992)

$$P(\eta\Delta) = \frac{1}{4\eta}\{2P(\Delta) + P(\Delta + 2) + P(\Delta - 2)\}. \tag{12}$$

Because the distribution function $P(\Delta)$ describes the distribution of, for example, nearest distances, excitation energies, etc, it is of physical importance. In the case of excitation energies, the corresponding $P(\Delta)$ determines the corresponding specific heat. Therefore, it is of primary interest to study the solutions of (12) as a function of η which is a parameter uniquely determined by the function V (Reichert and Schilling 1985). In line with our assumptions, *compactly supported continuous solutions* of (12) are of special interest.

1.2. Formulation of the main result

Rewrite (12) in the form

$$y(qt) = \frac{1}{4q}[y(t+1) + y(t-1) + 2y(t)] \qquad 0 < q < 1 \tag{13}$$

with η replaced by q in order to stress the analogy with q-difference equations theory. The existence and non-existence of continuous compactly supported solutions of (13) (as a function of q) was intensively studied in Baron (1988), Baron and Volkmann (1993), Baron *et al* (1994), Morawiec (1993) and Förg-Rob (1994).

Equation (13), rewritten in the form

$$y(t) = \lambda[\tfrac{1}{4}y(\lambda t - 1) + \tfrac{1}{2}y(\lambda t) + \tfrac{1}{4}y(\lambda t + 1)] \tag{14}$$

where $\lambda = 1/q$, is a special case of the two-scale difference equation or refinement equation

$$y(t) = \sum_{j=0}^{l} a_j y(\alpha t - \beta_j) \qquad \alpha > 1 \qquad \beta_j \in \mathbb{R} \tag{15}$$

studied in Daubechies and Lagarias (1991), Derfel *et al* (1995) and Derfel (1989).

An absolutely integrable function $y(t) \in L^1(-\infty, \infty)$ is called an L^1-solution of (15), if it satisfies (15) for almost all $t \in \mathbb{R}$.

Denote $\Delta = (1/\alpha)\sum_{j=0}^{l} a_j$ and suppose that $\Delta = 1$, which holds in case of (14). The following important result was proved in Daubechies and Lagarias (1991): provided $\Delta = 1$, any L^1-solution of (15) is compactly supported and unique (up to normalization).

A similar result concerning equation (13) was proved in Baron and Volkmann (1993).

The non-existence of compactly supported continuous solutions for 'small' values of q was repeatedly mentioned in the the literature. Such a non-existence result was proved for $0 < q < \frac{1}{3}$ by one of us, then for $0 < q < \sqrt{2} - 1 \approx 0.414$ by Baron (1994), for $0 < q < (1 - \sqrt[3]{2} + \sqrt[3]{4})/3 \approx 0.442$ by Morawiec (1993), and recently for $0 < q < \frac{1}{2}$ by Baron *et al* (1994).

On the other hand, except for the cases $q = 2^{-1/n}$, $n \in \mathbb{N}$, no non-trivial solutions of (13) have been known up till now.

(If $q = \frac{1}{2}$ then $y_{1/2}(t) = \max\{0, 1 - |t|\}$; this is the Schönberg B_1-spline. If $q = (\frac{1}{2})^{1/n}$, $n = 2, 3, \ldots$ then

$$y_{2^{-\frac{1}{k}}}(t) = B_1(t) * B_1(2^{-\frac{1}{k}}t) * \cdots * B_1(2^{-\frac{k-1}{k}}t)$$

(Baron *et al* 1994)).

The goal of this paper is to prove that:

(i) A continuous compactly supported solution of (13) exists for almost all $\frac{1}{2} \leqslant q < 1$.

Moreover, there exists a sequence $\beta_k \to 1$, such that for almost all $q \in (\beta_k, 1)$ equation (13) possesses a compactly supported solution with $2(k-1)$ derivatives.

(ii) The words for 'almost all' in the above statement cannot be omitted. There are exceptional values of q in the interval $\frac{1}{2} < q < 1$, for which no L_1-solution of (13) exists. For instance, $q_1 = \frac{\sqrt{5}-1}{2} \approx 0.618$ (the reciprocal of the 'golden ratio') and $q_2 \approx 0.755$ (the reciprocal of the positive root of the equation $x^3 - x - 1 = 0$) are such exceptional values.

More generally, if λ is any Pisot–Vijayaraghavan number, or any Salem number, then $q = \lambda^{-1}$ is an exceptional value.

Above result combined with the one of Baron *et al* (1994) shows that in the parameter interval $0 < q < 1$ the point $q = \frac{1}{2}$ defines the threshold between the existence and the non-existence of continuous compactly supported solutions of (13). This exhausts the problem of the existence of continuous compactly supported solutions of (13), in some sense.

The rest of the paper is organized as follows. Statements (i) and (ii) are proved in section 4 (theorem 5 and theorem 4, respectively). The proof is based on Erdös's (1939, 1940) results on infinite Bernoulli convolutions, and recent developments regarding Erdös's problem due to Solomyak (1995). Fourier analysis of (13) and its reduction to Erdös's problem is given in section 2.

In section 3 we obtain upper bounds for the smoothness of solutions of (13). Baron *et al*'s (1994) theorem on the non-existence of bounded, L^1-solutions, for $0 < q < 1$, is obtained as a consequence of these estimates.

When this work was close to completion an interesting related paper (Borwein and Girgensohn 1994) was brought to our attention by Professor K Baron. Borwein and Girgensohn's approach is similar to ours and, in particular, theorem 4(a) was proved in Borwein and Girgensohn (1994). However, our existence theorem is stronger than theirs: whereas in Borwein and Girgensohn (1994) the existence result is proved for almost all q close enough to 1 only, we prove this for almost all $\frac{1}{2} \leqslant q < 1$.

2. Fourier analysis and solutions for special values of q

In this section we shall study L^1-solutions of (13) by means of Fourier transform. A function $y(t) \in L^1(-\infty, \infty)$ is called an L^1-solution of (13), if it satisfies (13) for almost all $t \in \mathbb{R}$.

Denote the Fourier transform of a function $y(t) \in L^1(-\infty, \infty)$ by $f(p) = F[y(t)]$

$$f(p) = F[y(t)] = \int_{-\infty}^{\infty} y(t) \mathrm{e}^{\mathrm{i}tp}\, \mathrm{d}p$$

and the inverse Fourier transform by $F^{-1}(f(p)) = y(t)$

$$F^{-1}[f(p)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(p) \mathrm{e}^{-\mathrm{i}tp}\, \mathrm{d}p.$$

Theorem 1. (a) There exists, at most, one non-trivial L^1-solution $y(t)$ of (13) up to normalization.

(b) If such a solution exists, it has compact support and

$$\operatorname{supp} y(t) \subset \left[-\frac{q}{1-q}, \frac{q}{1-q} \right].$$

(c) The Fourier transform $F[y_q(t)] = f_q(p)$ of $y(t) = y_q(t)$ is given by

$$f(p) = f_q(p) = A \prod_{n=0}^{\infty} \cos^2(\tfrac{1}{2} q^n p) \tag{16}$$

where $A = f(0) = \int_{-\infty}^{\infty} y(t)\, \mathrm{d}t \neq 0$ and the infinite product converges for all p.

Proof. As was mentioned already parts (a) and (b) follow from Daubechies and Lagarias (1991) or Baron and Volkmann (1993). Part (c) can be checked by direct computation. In fact, applying Fourier transform to (13) and using the formula

$$F[y(\alpha t + \beta)] = \frac{1}{\alpha} \mathrm{e}^{\frac{-\mathrm{i}\beta p}{\alpha}} f(\frac{p}{\alpha}) \qquad \alpha, \beta \in \mathbb{R}$$

we obtain

$$\frac{1}{q} f\left(\frac{p}{q}\right) = \frac{1}{4q}(2\cos p + 2) f(p) = \frac{1}{q} \frac{1 + \cos p}{2} f(p)$$

or

$$f\left(\frac{p}{q}\right) = \left(\cos^2 \frac{p}{2}\right) f(p) \tag{17}$$

and finally

$$f_q(p) = f(p) = A \prod_{n=0}^{\infty} [\cos(\tfrac{1}{2} q^n p)]^2.$$

Denote by $f_q^0(p)$ the solution of (17) normalized by condition $f_q^0(0) = \int_{-\infty}^{\infty} y(t)\, \mathrm{d}t = 1$.

Corollary 1. (Garsia 1962, Baron*et al* 1994). A useful identity valid for all integers $k \geqslant 2$ follows from (17)

$$f_q^0(p) = f_{q^k}^0(p) f_{q^k}^0(qp) \cdots f_{q^k}^0(q^{k-1}p). \tag{18}$$

For $q = \frac{1}{2}$ equation (14) takes the form

$$y(t) = \tfrac{1}{2} y(2t-1) + y(2t) + \tfrac{1}{2} y(2t+1) \tag{19}$$

and its unique L^1-solution normalized under condition $\int_{-\infty}^{\infty} y(t)\, \mathrm{d}t = 1$ is the function $y_{\frac{1}{2}} = \max\{0, 1 - |t|\}$, i.e. Schönberg's B_1-spline (see figure 1(a)). Using that fact and (18) we obtain:

Corollary 2. (Garsia 1962, Baron *et al* 1994). For $q = (\frac{1}{2})^{1/k}$ the unique up to normalization solution of (13) is given by formula

$$y_{2^{-\frac{1}{k}}}(t) = B_1(t) * B_1(2^{-\frac{1}{k}}t) * \cdots * B_1(2^{-\frac{k-1}{k}}t).$$

This solution is $2(k-1)$ times differentiable. Figures 1(*b*) and (*c*) show $y_q(t)$ for $q = 2^{-1/2}$ and $q = 2^{-1/3}$.

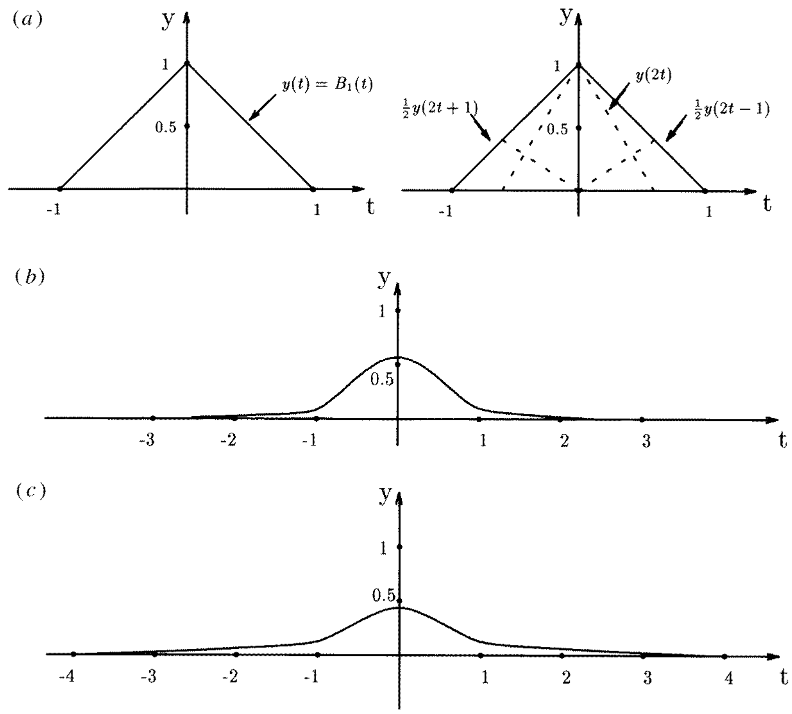

Figure 1. Solutions $y_q(t)$ of (13) for $q = 2^{-1/k}$, $k = 1, 2, 3$. (*a*) $q = 2^{-1}$, (*b*) $q = 2^{-1/2}$, (*c*) $q = 2^{-1/3}$.

Remark 1. The probabilistic meaning of (16) is transparent. Define two independent identically distributed (IID) random variables

$$Z_i = \tfrac{1}{2}(\eta_0^i + q\eta_1^i + \cdots + q^n\eta_n^i + \cdots) \qquad i = 1, 2 \tag{20}$$

where η_k^i are IID Bernoulli random variables which accept values $(+1)$ and (-1) with equal probabilities $\frac{1}{2}$. Define

$$Z = Z_1 + Z_2. \tag{21}$$

The characteristic function of Z_i $(i = 1, 2)$ is $\prod_{n=0}^{\infty}\cos(\frac{1}{2}q^n p)$ and the characteristic function of Z is $\prod_{n=0}^{\infty}\cos^2(\frac{1}{2}q^n p)$.

Hence $y(t)$ is the density function of Z if it exists.

Corollary 3. Suppose that $y(t)$ is a non-trivial L^1-solution of (13). Denote by $U_0(\varepsilon)$ an ε-neighbourhood of the point zero, and by $U_{-1}(\varepsilon)$ and $U_1(\varepsilon)$ right and left half-ε-neighbourhoods of the points $-\frac{q}{q-1}$ and $\frac{q}{q-1}$, respectively. Denote also

$$U_i^+(\varepsilon) = \{t \in U_i(\varepsilon) | y(t) > 0\} \qquad i = -1, 0, 1.$$

Then for any $\varepsilon > 0$ and $i = -1, 0, 1$ mes $(U_i^+(\varepsilon)) > 0$.

Proof. Denote by $\Phi_Z(t)$ the distribution function of Z and by P the corresponding probability measure: $P[a,b] = \Phi_Z(b) - \Phi_Z(a)$. From remark 1 it follows that $P(U_i(\varepsilon)) > 0$ for any $\varepsilon > 0$ and $i = -1, 0, 1$. On the other hand, the existence of a non-trivial L^1-solution of (13) means that $\Phi_Z(t)$ is absolutely continuous, with density function $y(t)$ and

$$P(U_i(\varepsilon)) = \int_{U_i(\varepsilon)} y(t)\,\mathrm{d}t.$$

Hence, for any $\varepsilon > 0 y(t)$, is positive on subsets of $U_i(\varepsilon)$, $i = -1, 0, 1$, of positive measure, that is mes $(U_i^+(\varepsilon)) > 0$.

3. Upper bounds for the smoothness of solutions and nonexistence of L^1-solutions for $0 < q < \frac{1}{2}$

We begin with a simple upper bound for the degree of smoothness of the solutions of (14).

Theorem 2. If $y(t)$ is a non-trivial, k times differentiable, compactly supported solution of (14) then

$$k \leqslant -1 + 2\frac{\ln 2}{\ln \lambda}. \tag{22}$$

Proof. According to Daubechies and Lagarias (1991) supp $y(t) \subset [-\frac{1}{\lambda-1}, \frac{1}{\lambda-1}]$. Define a new function $z(t)$ as a shift of $y(t)$: $y(t) = z(t - \frac{1}{\lambda-1})$ and denote $\tau = t - \frac{1}{\lambda-1}$. Then supp $z(\tau) \subset [0, \frac{2}{\lambda-1}]$ and $z(\tau)$ satisfies the equation

$$z(\tau) = \lambda[\tfrac{1}{4}z(\lambda\tau) + \tfrac{1}{2}z(\lambda\tau - 1) + \tfrac{1}{4}z(\lambda\tau - 2)]. \tag{23}$$

Near the origin ($\tau = 0$) the second and third terms on the right-hand side of (23) vanish: $z(\lambda\tau - 1) = 0$ and $z(\lambda\tau - 2) = 0$, and (23) reduces to a two-term equation

$$z(\tau) = \frac{\lambda}{4} z(\lambda\tau). \tag{24}$$

The general solution of the two-term functional equation

$$z(\tau) = a z(\alpha\tau) \tag{25}$$

is well known (Peluch and Sharkovsky 1974):

$$z(\tau) = |\tau|^\gamma \begin{cases} K_+\left(\dfrac{\ln|\tau|}{\ln|\alpha|}\right) & \tau > 0 \\ K_-\left(\dfrac{\ln|\tau|}{\ln|\alpha|}\right) & \tau < 0 \end{cases} \tag{26}$$

where K_+, K_- are arbitrary one-periodic functions and

$$\gamma = -\frac{\ln|a|}{\ln|\alpha|}. \tag{27}$$

In particular, for (24)

$$\gamma = -\frac{\ln\frac{\lambda}{4}}{\ln\lambda} = -1 + 2\frac{\ln 2}{\ln\lambda}. \tag{28}$$

If $z(\tau) \not\equiv 0$, then for any sufficiently small $\varepsilon > 0$, according to corollary 3, $z(\tau) \not\equiv 0$ for $0 < \tau < \varepsilon$ and thus $K_+ \not\equiv 0$ for $0 < t < \varepsilon$.

It follows then from (26) that the degree of smoothness of $z(\tau)$ is no more than γ and (28) completes the proof.

In a similar manner, the following result can be proved in the case $0 < q < \frac{1}{2}$.

Theorem 3. (Baron *et al* 1994) If $0 < q < \frac{1}{2}$, then (14) does not possess any non-trivial, bounded, L^1-solution. (In particular, it does not possess a non-trivial, continuous, compactly supported solution.)

Proof. If $0 < q < \frac{1}{2}$, then $\lambda > 2$ and $\operatorname{supp} y(t) \subset [-\frac{1}{\lambda-1}, \frac{1}{\lambda-1}] \subset [-1, 1]$. Take $|t| < \frac{1}{\lambda}(1 - \frac{1}{\lambda-1})$ (such a t exists because $1 - \frac{1}{\lambda-1} > 0$, for $\lambda > 2$). Then

$$|\lambda t \pm 1| \geqslant 1 - \left(1 - \frac{1}{\lambda - 1}\right) = \frac{1}{\lambda - 1}$$

and (14) reduces to the two-term equation $y(t) = \frac{\lambda}{2} y(\lambda(t))$, a general solution of which is given by formula (26) with

$$\gamma = -\frac{\ln \lambda/2}{\ln \lambda} = -1 + \frac{\ln 2}{\ln \lambda}$$

under the assumption of the theorem $\lambda > 2$ and thus $-1 < \gamma < 0$. In addition, if $y(t)$ is a non-trivial L^1-solution of (14), then, for any sufficiently small $\varepsilon > 0$, according to corollary 3, $y(t) > 0$ on the set $U_0^+(\varepsilon)$ of positive measure. Thus K_+ or $K_- \not\equiv 0$ on $U_0^+(\varepsilon)$. It follows now from (26) that $y(t)$ is unbounded on $U_0^+(\varepsilon)$. This contradiction completes the proof.

Remark 2. Compare the results of theorem 2 and corollary 2 in the case $q = (\frac{1}{2})^{1/k}$ (i.e. $\lambda = 2^{1/k}$). According to corollary 2 $y_{2^{-1/k}}(t)$ is $2(k-1)$ times differentiable, whereas, according to theorem 2, its degree of smoothness is at most

$$-1 + \frac{2 \ln 2}{\ln(2^{1/k})} = 2k - 1.$$

This means that the bound (22) cannot essentially be improved.

4. Existence of continuous compactly supported solutions for $\frac{1}{2} \leqslant q < 1$ and their smoothness

Looking at corollary 2 one might be inclined to expect that a continuous, compactly supported solution of (13) exists for all $q \in [\frac{1}{2}, 1)$ and its smoothness cannot decrease when q is increasing. However, it turns out that this is not the case. There exist exceptional values of $q \in [\frac{1}{2}, 1)$ for which continuous, compactly supported solutions of (13) do not exist.

These exceptional values of q are reciprocal to Pisot–Vijayaraghavan numbers (PV numbers).

Definitions. A *PV number* is a real algebraic integer (that is a root of a polynomial with integer coefficients and the coefficient of the highest power is unity) which is greater than 1 and all of whose conjugates have absolute value less than 1.

For example, the polynomial $x^2 - x - 1$ has roots $x_{1,2} = \frac{1}{2}(1 \pm \sqrt{5})$. The algebraic integer $x_1 = \frac{1}{2}(1 + \sqrt{5})$ is greater than 1 and its conjugate $x_2 = \frac{1}{2}(1 - \sqrt{5})$ is less than 1. Hence the 'golden ratio' $\varphi = \frac{1}{2}(1 + \sqrt{5})$ is a PV number, and its reciprocal $q = \frac{1}{\varphi} = \frac{\sqrt{5}-1}{2} = 0.618$ is an 'exceptional value'. It is known (Siegel 1944) that the smallest PV number $x_3 = 1.324\ldots$ is the positive root x_3 of the equation $x^3 - x - 1 = 0$, and its reciprocal $q = 1/x_3 \approx 0.755$ is an 'exceptional value'.

A *Salem number* is a real algebraic integer greater than 1, whose other conjugates have modulus at most equal to 1, with at least one having a modulus equal to one. There are no examples of Salem numbers as simple as the ones given for PV numbers, because there exist no Salem numbers of degree less than 4. It is an open question whether there is a sequence of Salem numbers, which tends to 1, or on the contrary, whether the set T of Salem numbers is closed on the real line. The smallest known Salem number is 1.176..., with its reciprocal $q = 0.850\ldots$ (Boyd 1977).

Theorem 4. (a) For all q reciprocal to a PV number equation (13) does not possess a compactly supported continuous solution. (Moreover, it does not even possess an L^1-solution for these values of q.)

(b) For all q reciprocal to a Salem number equation (13) does not possess a compactly supported continuous solution.

Proof. (a) For any L^1-solution $y_q(t)$ of (13) its Fourier transform is given by formula

$$f_q(p) = A\left[\prod_{n=0}^{\infty}\cos(\tfrac{1}{2}q^n p)\right]^2$$

and tends to zero as $p \to \infty$, according to the Riemann–Lebesque lemma. On the other hand, for any q reciprocal to a PV number Erdös (1939) proved that

$$l_q(p) = \prod_{n=0}^{\infty}\cos(\tfrac{1}{2}q^n p)$$

does not tend to zero, as $p \to \infty$. This contradiction proves part (a).

(b) Suppose a continuous compactly supported solution $y_q(t)$ of (13) exists for q reciprocal to a Salem number. Clearly that $y_q(t)$ belongs to both spaces $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$ together. Therefore the L^2-Fourier transform of $y_q(t)$ belongs to $L^2(\mathbb{R})$ and coincides with its usual L^1-Fourier transform $F[y_q(t] = f_q(p)$. Thus $f_q(p) \in L^2(\mathbb{R})$.

On the other hand, it is known (Kahane 1971) that for any $\varepsilon > 0$ and some $C > 0$

$$|l_q(p)| > \frac{C}{p^\varepsilon}$$

if q is reciprocal to a Salem number. Hence

$$|f_q(p)| > \frac{C}{p^{2\varepsilon}}$$

and $f_q(p) \notin L^2(\mathbb{R})$. This contradiction proves the theorem.

Remark 3. It is known (Bertin *et al* 1992) that there are infinitely many PV numbers between 1 and 2 and therefore there are infinitely many exceptional values of q between $\frac{1}{2}$ and 1.

Although there exist exceptional values of q in $(\frac{1}{2}, 1)$ the following is true.

Theorem 5. (a) For almost all $\frac{1}{2} \leqslant q < 1$, a continuous compactly supported solution of (13) exists.

(b) Moreover, there exists a sequence $\beta_k \in (1/2, 1)$, $\beta_k \to 1$, such that for almost all $q \in (\beta_k, 1)$, (13) possesses a compactly supported solution with $2(k-1)$ derivatives.

Proof. Part (b) is a consequence of Erdös's result (Erdös 1940), who proved that for any positive integer k there exists a sequence $\beta_k \to 1$, such that the set of points q of the interval $(\beta_k, 1)$ for which

$$l_q(p) = \mathrm{o}(|p|^{-k}) \qquad p \to \infty$$

does not hold, is a set of measure zero.

(a) Solomyak (1995) has proved recently that in fact $l_q(p) \in L^2(\mathbb{R})$ for almost all $q \in (\frac{1}{2}, 1)$. Then, according to (17) $f_q(p) = A[l_q(p)]^2 \in L^1(\mathbb{R})$ and thus $y_q(t) = F^{-1}[f_q(p)]$ is a continuous compactly supported solution of (13).

5. Conclusions

Let us conclude with a short discussion of the possible consequences of our results for the physical properties of disordered systems. First of all our results show that there is a qualitative difference between the range $0 < q < \frac{1}{2}$ and $\frac{1}{2} < q < 1$. For the former case the integral distribution function is generically singular continuous, e.g. it has a Cantor set as support. In case where we consider the distribution of low energy excitations this implies that the specific heat as a function of the temperature may possess a hierarchy of maxima, also called Schottky anomalies. For the latter case it is generically absolutely continuous. This usually implies (again for the specific heat $c(T)$) a smooth temperature dependence of $c(T)$ with one maximum, e.g. for the special value $q = \frac{1}{2}$ the exact solution of the functional equation is $y_{1/2}(t) = \max\{0, 1 - |t|\}$ which yields

$$c(T) \approx aT - bT^2 \qquad a > 0,\ b > 0$$

for a temperature range where the right-hand side of the last formula is sufficiently positive. The role of the non-generic values of q for which absolute continuity does not hold in this latter range is not quite obvious, because it is probably not possible to realize such special non-generic values in an experiment. Let us finally mention a more direct method to explore e.g. the distribution of excitation energies, which is a scattering experiment (neutrons for instance). Here it would be interesting to investigate the qualitative structure of the corresponding density function, also called density of states (DOS), for different types of disordered systems. Unfortunately, such an experiment cannot distinguish between singular continuous and absolutely continuous distribution functions due to finite resolution, but nevertheless the mathematically singular continuous case should manifest itself in a rather 'bizarre' DOS whereas the absolutely continuous case should yield a rather smooth DOS behaviour.

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References

Arnold V I and Avez A 1968 *Ergodic Problems in Classical Mechanics* (New York: Benjamin)

Baron K 1988 On a problem of R Schilling *Ber. Math.-Statist. Sekt. Forschungsge sellsch. Joanneum* **286** 1–6

Baron K and Volkmann P 1993 Unicité pour une équation fonctionelle *Ann. École Norm. Sup. Cracovie, 159, Travaux Math.* **13** 53–6

Baron K, Simon A and Volkman P 1994 Solutions d'une equation fonctionelle dans l'espace des distributions temperees *C. R. Acad. Sci., Paris* **319** 1249–52

Bertin M *et al* 1992 *Pisot and Salem Numbers* (Basel: Birkhäuser)

Borwein J M and Girgensohn R 1994 Functional equations and distribution functions *Res. Math.* **26** 229–37

Boyd D W 1977 Small Salem numbers *Duke Math. J.* **44** 315–28

Daubechies I and Lagarias J 1991 Two-scale difference equations, I. Existence and global regularity of solutions *SIAM J. Math. Anal.* **22** 1388–410

Derfel G 1989 Probabilistic method for a class of functional-differential equations *Ukrainian Math. J.* **41** 1137–41

Derfel G, Dyn N and Levin D 1995 Generalized refinement equations and subdivision processes *J. Approx. Theory* **80** 272–97

Erdös P 1939 On a family of symmetric Bernoulli convolutions *Am. J. Math.* **61** 974–6

——1940 On the smoothness properties of a family of Bernoulli convolutions *Am. J. Math.* **62** 180–6

Förg-Rob W 1994 On a problem of R Schilling, part I *Math. Pannonica* **5** 29–65; 1994 part II *Math. Pannonica* **5** 145–68

Garsia A 1962 Arithmetic properties of Bernoulli convolutions *Trans. Am. Math. Soc.* **102** 409–32

Kahane J P 1971 Sur la distribution de certaines series aleatoires *Bull. Soc. Math. France* **25** 119–22

Morawiec J 1993 On a problem of R Schilling *Proc. 3rd Int. Conf. on functional equations and inequalities (Koninki, 1991)* (Cracovie: Wyzsza Szkola Pedagogiczna, Inst. Math.) p 22

——1995 On continuous solution of a problem of R Schilling *Res. Math.* **27** 381–6

——On bounded solutions of a problem of R Schilling *Ann. Math. Sil.* to appear

Moser J 1973 *Stable and Random Motions in Nonlinear Dynamics* (Princeton, NJ: Princeton University Press)

Peluh G and Sharkovsky A 1974 *Introduction to the Theory of Functional Equations* (Kiev: Naukova dumka) (in Russian)

Reichert P and Schilling R 1985 Glasslike properties of a chain of particles with anharmonic and competing interactions *Phys. Rev.* B **32** 5731–46

Rubin B 1995 Private communication

Ruelle D and Takens F 1971 On the nature of turbulence *Commun. Math. Phys.* **20** 167–92

Schilling R 1992 Spatially-chaotic structures *Nonlinear Dynamics of Solid* ed E Thomas (Berlin: Springer) pp 213–41

Siegel C L 1944 Algebraic integers whose conjugates lie in the unit circle *Duke Math. J.* **11** 597–602

Solomyak B 1995 On an Erdös problem *Ann. Math.* **142** 611–25